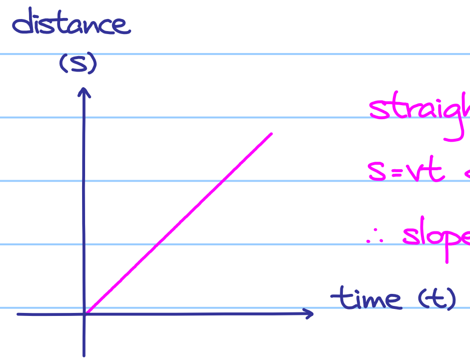
§ 5 Differentiation

5.1 Idea of Derivative

Recall : (average) speed = $\frac{\text{distance}}{\text{time}}$



straight line

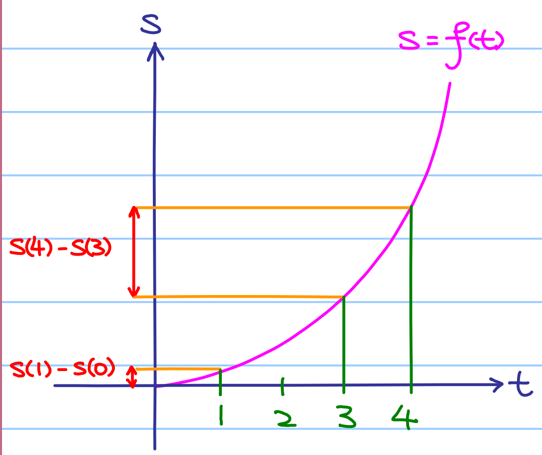
$s = vt \Leftrightarrow v = \frac{s}{t}$

$\therefore$ slope = v = speed .

Remark :

Using displacement and velocity if you know .

How about this case ?



distance traveled from t=0 to t=1 (S(1) - S(0)) < distance traveled from t=3 to t=4 (S(4) - S(3))

Why? The speed is changing.

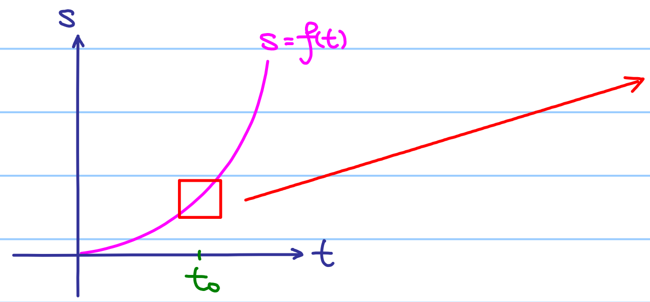
Speed is different at different moment.

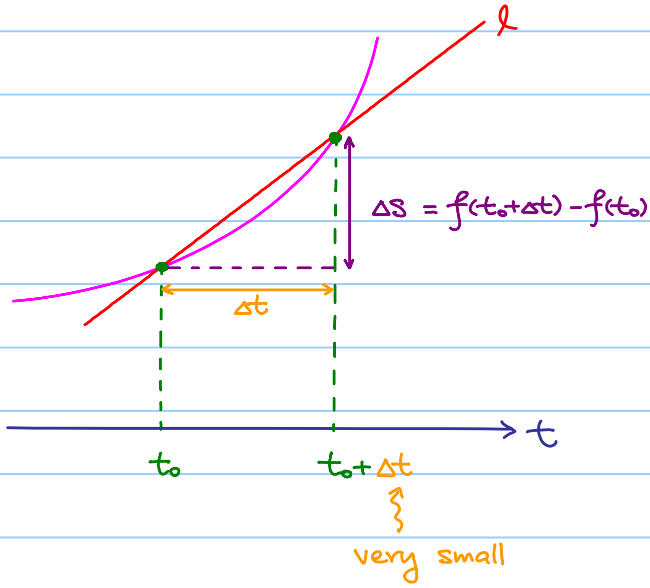
Hold on !

What is the meaning of speed at a particular moment (instantaneous speed) ?

We need a definition !

Instantaneous speed at $t = t_0$:





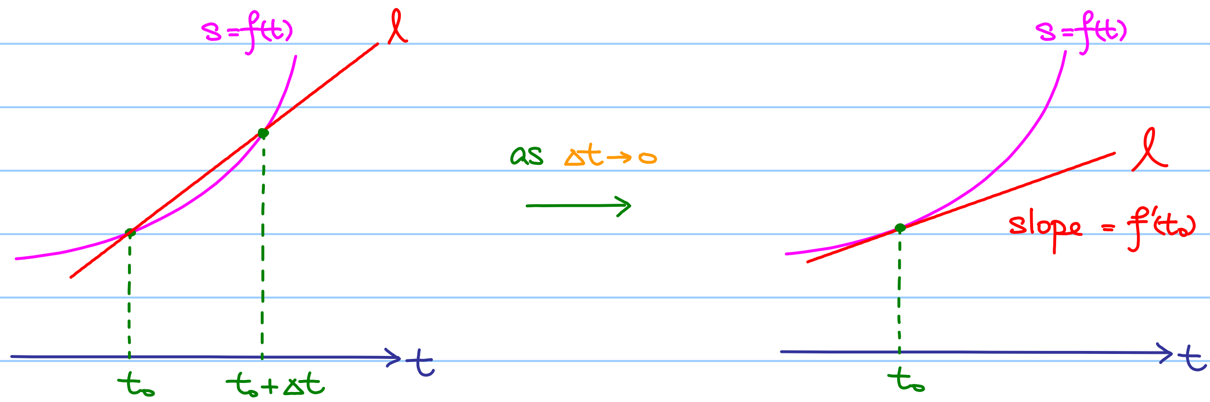
Average speed between t_0 and $t_0 + \Delta t$

$$= \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \text{slope of } \ell$$

Idea : Let Δt becomes smaller and smaller !

Instantaneous speed at $t = t_0$ is defined to be $\displaystyle\lim_{\Delta t \to 0} \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t}$

(provided it exists , if so , it is denoted by $f'(t_0)$)



Note : When $\Delta t \to 0$, ℓ becomes the tangent line at $t = t_0$, so

slope of the tangent line at $t = t_0$ $= f'(t_0)$

Example 5.1.1

If $s = f(t) = t^2$, find $f'(2)$ (instantaneous speed at $t=2$).

$$f'(2) = \lim_{\Delta t \to 0} \frac{f(2+\Delta t) - f(2)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{(2+\Delta t)^2 - 2^2}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{2 \cdot 2\Delta t + \Delta t^2}{\Delta t}$$
$$= \lim_{\Delta t \to 0} 2 \cdot 2 + \Delta t = 2 \cdot 2 = 4$$

In general, we have $y = f(x)$, fix x_0.

Then $f'(x_0)$ means rate of change of y with respect to x at $x = x_0$.

Definition 5.1.1

$f(x)$ is said to be differentiable at $x = x_0$ if $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ exists.

It is called the derivative of $f(x)$ at $x = x_0$ and it is denoted by $f'(x_0)$.

Note: By definition, if $f(x_0)$ is NOT well-defined, then $f'(x_0)$ is NOT well-defined.

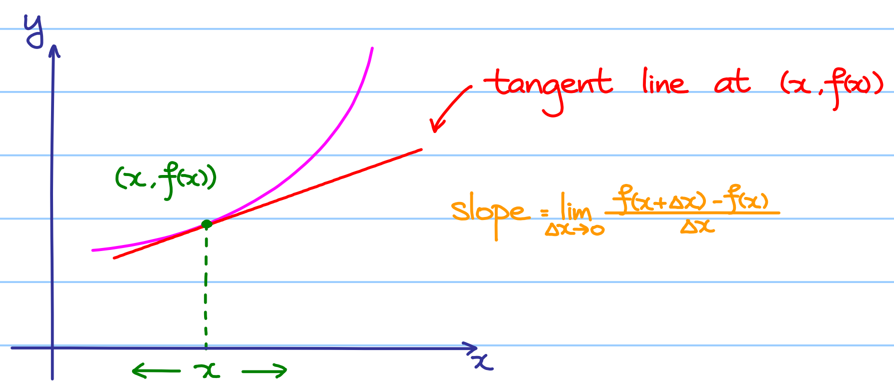
Let $\Delta x = x - x_0$, i.e. $x = x_0 + \Delta x$

When Δx tends to 0, x tends to x_0.

Therefore, we have another formulation:

$f(x)$ is said to be differentiable at $x = x_0$ if $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.

Perform the previous step at every point:



Recall: What is a function?

Roughly speaking, given an input x, return a value.

Now, we construct a new function, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ (if exists)

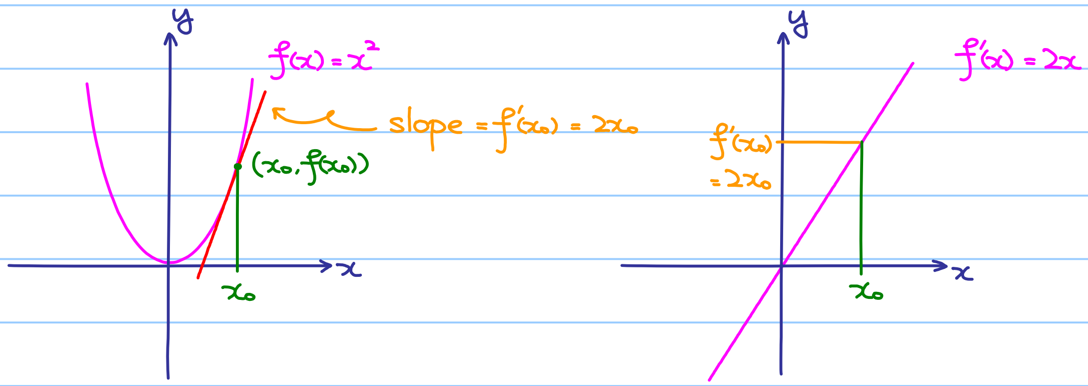
(i.e. given an input x, return the slope of the tangent line at $(x, f(x))$.)

Example 5.1.2

If $f(x) = x^2$, find $f'(x)$.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \Delta x = 2x
\end{aligned}
$$

Relation between the graphs of $f(x) = x^2$ and $f'(x) = 2x$:



Notations:

$$y = f(x) = x^2$$

$$\frac{df}{dx} = \frac{dy}{dx} = f'(x) = 2x$$

$$\left.\frac{df}{dx}\right|_{x=3} = \left.\frac{dy}{dx}\right|_{x=3} = f'(3) = 2(3) = 6$$

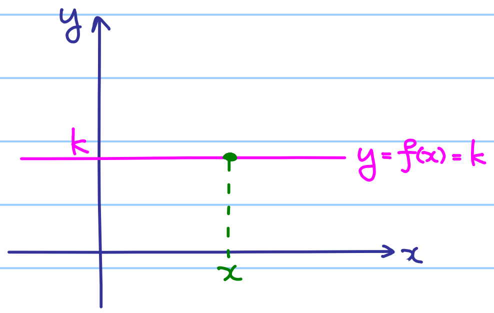
Definition 5.1.2

If $f: A \to \mathbb{R}$ is a function that is differentiable at every point in A,

then $f(x)$ is said to be a differentiable function.

Theorem 5.1.1

If $f(x) = k$, where k is a constant, then $f'(x) = 0$.



Note : tangent line at $(x, f(x))$ is horizontal

$\therefore f'(x) = 0$

Concrete computation :

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{k-k}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0$$

$$= 0$$

Exercise 5.1.1

Find $f'(x)$ if

(a) $f(x) = x$ Ans : $f'(x) = 1$

(b) $f(x) = x^3$ $f'(x) = 3x^2$

Theorem 5.1.2

If $f(x) = x^r$, where r is a real number, then $f'(x) = rx^{r-1}$ whenever it is defined.

(Think: If $r = \frac{1}{2}$, $f(x) = \sqrt{x}$ which is defined when $x \geq 0$.)

proof:

We only prove the case $f(x) = x^n$, where n is a natural number.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(C_0^n x^n + C_1^n x^{n-1} \Delta x + C_2^n x^{n-2} \Delta x^2 + \cdots + C_n^n \Delta x^n) - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} C_1^n x^{n-1} + \underbrace{C_2^n x^{n-2} \Delta x + \cdots + C_n^n \Delta x^{n-1}}_{\text{terms with powers of } \Delta x}$$

$$= nx^{n-1}$$

5.2 Differentiability and Continuity

Theorem 5.2.1

If $f(x)$ is differentiable at $x=x_0$, then $f(x)$ is continuous at $x=x_0$.

proof : By assumption, $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ exists

Also, we know $\lim_{\Delta x \to 0} \Delta x = 0$

$$\lim_{\Delta x \to 0} f(x_0+\Delta x)-f(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \cdot \Delta x$$

both exist

$$= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x$$

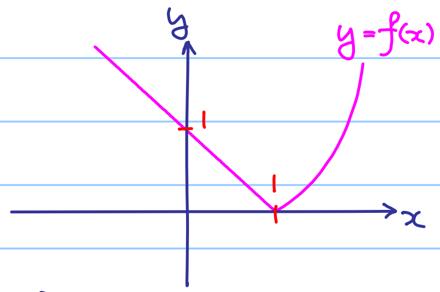
$$= f'(x_0) \cdot 0 = 0$$

$\therefore \lim_{\Delta x \to 0} f(x_0+\Delta x) = f(x_0)$, so $f(x)$ is continuous at $x=x_0$.

However, the converse is NOT true.

Example 5.2.1

Let $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$



$$\lim_{\Delta x \to 0^+} \frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{[(1+\Delta x)^2-1]-[1^2-1]}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{2\Delta x+\Delta x^2}{\Delta x} = 2$$

(it means we are looking at small but positive Δx)

$$\lim_{\Delta x \to 0^-} \frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{[1-(1+\Delta x)]-[1^2-1]}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

(it means we are looking at small but negative Δx)

$$\lim_{\Delta x \to 0^+} \frac{f(1+\Delta x)-f(1)}{\Delta x} \neq \lim_{\Delta x \to 0^-} \frac{f(1+\Delta x)-f(1)}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} \frac{f(1+\Delta x)-f(1)}{\Delta x} \text{ does NOT exist !}$$

$\therefore f(x)$ is NOT differentiable at $x=1$.

Exercise 5.2.1

a) Show that $f(x)$ is continuous at $x=1$, i.e. $\lim_{x\to 1} f(x) = f(1)$.

(Therefore, the converse statement of theorem 5.2.1 is NOT true.)

b) Write down $f'(x)$ for $x \neq 1$.

Answer:

$$f'(x) = \begin{cases} 2x & \text{if } x > 1 \\ \text{undefined} & \text{if } x = 1 \\ -1 & \text{if } x < 1 \end{cases}$$

5.3 Elementary Rules of Differentiation

Theorem 5.3.1

If $f(x)$ and $g(x)$ are differentiable functions, then

1) $(f+g)'(x) = f'(x) + g'(x)$

2) $(f-g)'(x) = f'(x) - g'(x)$

3) [product rule] $(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

4) [quotient rule] $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$

proof of (3):

$$\lim_{\Delta x\to 0} \frac{(f\cdot g)(x+\Delta x) - (f\cdot g)(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \underset{+}{g(x+\Delta x)} + f(x)\frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x)g(x) + f(x)g'(x)$$

$+$ $g(x)$ is differentiable

$\Rightarrow$ $g(x)$ is continuous

$\Rightarrow$ $\lim_{\Delta x\to 0} g(x+\Delta x) = g(x)$

Direct consequence:

Theorem 5.3.2

If k is a constant and $f(x)$ is a differentiable function, then $(k\cdot f)'(x) = kf'(x)$.

proof:

Using the product rule and theorem 5.1.1

$$(k\cdot f)'(x) = \underbrace{(k)'}_{0} f(x) + kf'(x) = kf'(x)$$

Example 5.3.1

Find $\frac{d}{dx}(3x^2+7x-2)$

$$\begin{aligned}\frac{d}{dx}(3x^2+7x-2) &= \frac{d}{dx}(3x^2)+\frac{d}{dx}(7x)-\frac{d}{dx}(2)\\ &= 3\frac{d}{dx}(x^2)+7\frac{d}{dx}(x)-\frac{d}{dx}(2)\\ &= 3(2x)+7(1)-0\\ &= 6x+7\end{aligned}$$

Example 5.3.2

Find $\frac{d}{dx}(3x^2-5x+1)(2x+7)$

$$\begin{aligned}&\frac{d}{dx}[(3x^2-5x+1)(2x+7)]\\ &= \left[\frac{d}{dx}(3x^2-5x+1)\right](2x+7)+(3x^2-5x+1)\left[\frac{d}{dx}(2x+7)\right]\\ &= (6x-5)(2x+7)+(3x^2-5x+1)(2)\\ &= 18x^2+22x-33\end{aligned}$$

Try to compare : Expand $(3x^2-5x+1)(2x+7)$ and get $6x^3+11x^2-33x+7$

Then differentiate , get the same result ?

Example 5.3.3

Find the derivative of $\frac{2x}{x^2+1}$.

$$\begin{aligned}\frac{d}{dx}\frac{2x}{x^2+1} &= \frac{\left[\frac{d}{dx}(2x)\right](x^2+1)-(2x)\left[\frac{d}{dx}(x^2+1)\right]}{(x^2+1)^2}\\ &= \frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}\\ &= \frac{-2x^2+2}{(x^2+1)^2}\end{aligned}$$

Example 5.3.4

Find the derivative of $\frac{1}{\sqrt{x}}+\sqrt{x}$

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right) &= \frac{d}{dx}\left(x^{-\frac{1}{2}}+x^{\frac{1}{2}}\right)\\ &= -\frac{1}{2}x^{-\frac{3}{2}}+\frac{1}{2}x^{-\frac{1}{2}}\end{aligned}$$

5.4 Higher Derivatives

$s(t)$: distance functions (depends on time t)

(instantaneous) speed = rate of change of distance travelled with respect to t.

$$v(t) = \frac{ds}{dt} \quad \text{(still a function of } t\text{)}$$

Question : What is $\frac{dv}{dt}$?

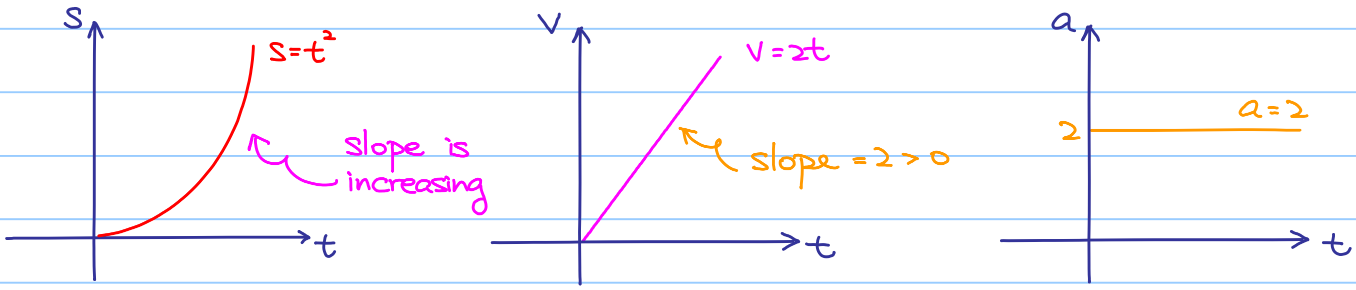
Answer : Acceleration !

= rate of change of speed with respect to t.

We write $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Example 5.4.1

$s(t) = t^2$ $\qquad v(t) = \frac{ds}{dt} = 2t$ $\qquad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 2$



speed is increasing

i.e. accelerating

Notations :

In general, let $y = f(x)$.

We have : (1st derivative) $\frac{dy}{dx} = \frac{df}{dx} = f'(x)$

(2nd derivative) $\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x)$

(n th derivative) $\frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = f^{(n)}(x)$

5.5 Derivatives of Trigonometric Functions

Preparations :

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2(\frac{x}{2})}{x^2}$$

Note: $\cos x = 1-2\sin^2(\frac{x}{2})$

$\therefore\ 1-\cos x = 2\sin^2(\frac{x}{2})$

$$= \lim_{x\to 0}\frac{1}{2}\frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2}$$

$$= \frac{1}{2}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot x$$

$$= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\lim_{x\to 0}x$$

$$= \frac{1}{2}\cdot 0$$

$$= 0$$

$$\lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \lim_{\Delta x\to 0}\frac{\cos x\cos\Delta x-\sin x\sin\Delta x-\cos x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\cos x\,\frac{\cos\Delta x-1}{\Delta x}-\sin x\,\frac{\sin\Delta x}{\Delta x}$$

$$= -\sin x$$

$(\because\ \lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x}=0$ and $\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=1.)$

$$\therefore\ \frac{d}{dx}\cos x = -\sin x$$

Exercise 5.5.1

Show that $\frac{d}{dx}\sin x = \cos x$ by using method similar to the above.

$\tan x = \frac{\sin x}{\cos x}$ $\quad \sec x = \frac{1}{\cos x}$ $\quad \csc x = \frac{1}{\sin x}$ $\quad \cot x = \frac{\cos x}{\sin x}$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} \overset{(*)}{=} \cdots = \frac{1}{\cos^2 x} = \sec^2 x$$

(*) Exercise : By quotient rule

Exercise 5.5.2

Show that

a) $\frac{d}{dx}\sec x = \sec x\tan x$

b) $\frac{d}{dx}\csc x = -\csc x\cot x$

c) $\frac{d}{dx}\cot x = -\csc^2 x$

5.6 Derivative of e^x

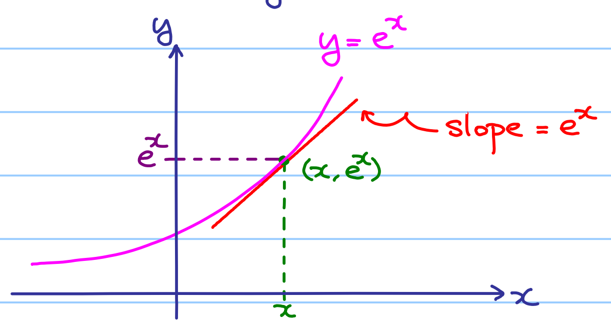
Cheating : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\frac{d}{dx} e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= e^x \qquad \text{(getting back itself)}$$

Geometrical meaning :



Example 5.6.1

Find $\frac{d}{dx}[e^x(3x^2 + 7x - 2)]$

$$\frac{d}{dx}[e^x(3x^2+7x-2)] = \left[\frac{d}{dx}e^x\right](3x^2+7x-2) + e^x\left[\frac{d}{dx}(3x^2+7x-2)\right]$$

$$= e^x(3x^2+7x-2) + e^x(6x+7)$$

$$= e^x(3x^2+13x+5)$$

Question : How do we differentiate a more complicated function , such as $\sqrt{x^2+3x}$?

We need a tool called chain rule .

5.7 Chain Rule

Theorem 5.7.1

If $f: B \to \mathbb{R}$ and $g: A \to \mathbb{R}$ are differentiable functions such that $g(A) \subseteq B$, then the composite function $(f \circ g): A \to \mathbb{R}$ defined by $(f \circ g)(x) = f(g(x))$ is differentiable and $(f \circ g)'(x) = f'(g(x))\, g'(x)$.

Hard to understand ? Let's reformulate it as :

Let $u = g(x)$, $y = f(u) = f(g(x))$, then

the chain rule simply means $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Think : $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$

Example 5.7.1

Find the derivative of $\sqrt{x^2+3x}$.

Let $u = g(x) = x^2+3x$, $\qquad \frac{du}{dx} = 2x+3$

$y = f(u) = \sqrt{u}$ $\qquad \frac{dy}{du} = \frac{1}{2\sqrt{u}}$

then $f(g(x)) = \sqrt{x^2+3x}$

By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \cdot (2x+3)$

$= \underbrace{\frac{1}{2\sqrt{x^2+3x}}}_{f'(g(x))} \cdot \underbrace{(2x+3)}_{g'(x)}$ put $u = x^2+3x$ back

differentiate f
then put back $g(x)$

Example 5.7.2

Find the derivative of $(3x^2-2x)^{2016}$.

Let $u = g(x) = 3x^2-2x$ $\qquad \frac{du}{dx} = 6x-2$

$y = f(u) = u^{2016}$ $\qquad \frac{dy}{du} = 2016\,u^{2015}$

then $f(g(x)) = (3x^2-2x)^{2016}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 2016\,u^{2015} \cdot (6x-2)$

$= 2016(3x^2-2x)^{2015} \cdot (6x-2)$ put $u = 3x^2-2x$ back

$= 4032(3x^2-2x)^{2015} \cdot (3x-1)$

Slogan : differentiate layer by layer.

Exercise 5.7.1

Show that $\frac{d}{dx} e^{ax} = a e^{ax}$

Exercise 5.7.2

Find the derivative of $\left(\frac{x}{x+1}\right)^2$

a) by using the chain rule ;

b) by writing $\left(\frac{x}{x+1}\right)^2 = \frac{x^2}{(x+1)^2}$ and using the quotient rule.

Answer : Both equal to $\frac{2x}{(x+1)^3}$.

Example 5.7.3

Find the derivative of $e^{\sqrt{x^2+1}}$.

1st layer $y = e^w$ $\quad w = \sqrt{x^2+1}$

2nd layer $w = \sqrt{u}$ $\quad u = x^2+1$

3rd layer $u = x^2+1$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

$$= e^{\sqrt{x^2+1}} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$= \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

Example 5.7.3

Revisit of quotient rule :

$$\left(\frac{f}{g}\right)'(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\,[g(x)]^{-1}\right)$$

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\frac{d}{dx}[g(x)]^{-1}$$

Apply the chain rule

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\left\{-[g(x)]^{-2}\frac{dg}{dx}\right\}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

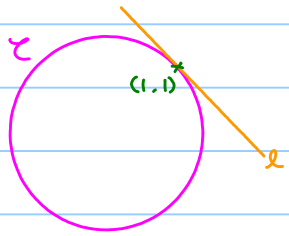
5.8 Implicit Differentiation

Example 5.8.1

$x^2+y^2=2$ —— $\mathcal{C}$

Locus of $\mathcal{C}$ is a circle centered at $(0,0)$ with radius $\sqrt{2}$.

Check: $(1,1)$ is a point lying on the circle.



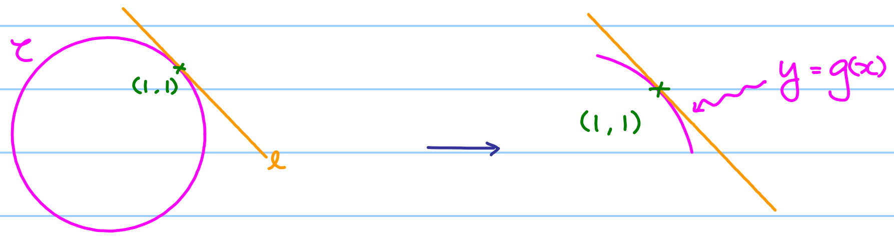
We want to find the equation of the tangent line ℓ

(i.e. need to know the slope of ℓ)

Note: $x^2+y^2=2$ is NOT a function.

Question: How to find $\frac{dy}{dx}$? (and, actually, is it defined?)

Answer: Yes, roughly speaking,



The small segment of $\mathcal{C}$ containing $(1,1)$ can be regarded as the graph of some function $y=g(x)$. (In fact, $y=\sqrt{2-x^2}$ in this case.)

How to find? Do it as usual!

$$x^2+y^2=2$$

differentiate both sides with respect to x.

$$2x+\frac{d}{dx}y^2=0$$

$$2x+2y\frac{dy}{dx}=0 \qquad \text{(Applying chain rule)}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

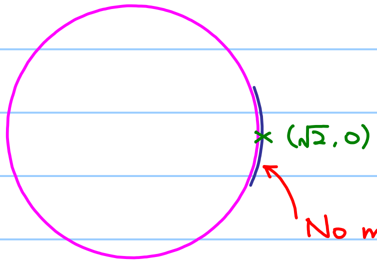
$\therefore \frac{dy}{dx}=-1$ when $(x,y)=(1,1)$.

We denote it by $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}=-1$

Remark :

$\frac{dy}{dx}$ is defined at a point of a curve only if a small arc containing the point can be regarded as the graph of some function $y=g(x)$.

$\therefore$ $\frac{dy}{dx}$ is NOT defined when $(x,y)=(\pm\sqrt{2},0)$.



No matter how small the arc is,

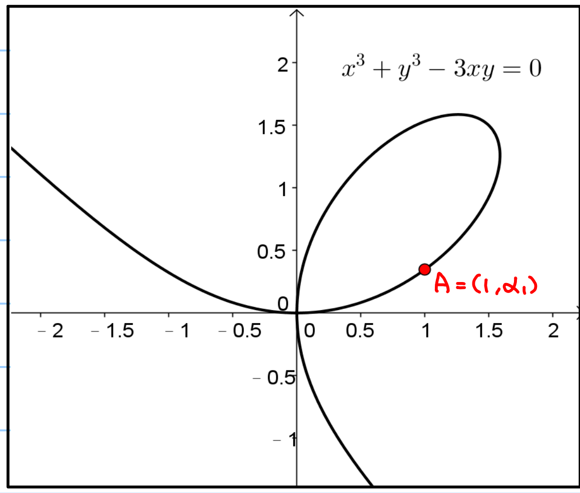
it cannot be realized as graph of some function $y=g(x)$.

Example 5.8.2

$$x^3+y^3-3xy=0 \quad \text{—— } \mathcal{C}$$

$$3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y-x^2}{y^2-x}$$



If we want to find the slope of the tangent line at A.

putting $x=1$ into $\mathcal{C}$.

$y^3-3y+1=0$

NOT easy to solve !

FACT : The above equation has three roots, two roots α_1, α_2 are positive $(\alpha_1<\alpha_2)$

one root is negative.

$A=(1,\alpha_1)$ and what we need is $\left.\frac{dy}{dx}\right|_{(x,y)=(1,\alpha_1)}$

Applications :

Example 5.8.3

Differentiation of Logarithmic Function

Let $y = \ln x$, $x > 0$. Then $e^y = x$,

differentiate both sides with respect to x .

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln x = \frac{1}{x}$ for $x > 0$.

Exercise 5.8.1

By rewriting $\log_a x = \frac{\ln x}{\ln a}$, show that $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$.

Example 5.8.4

Let $y = \ln|x|$, $x \neq 0$. Find $\frac{dy}{dx}$.

We can rewrite $y = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$

For $x > 0$, we have just shown that $\frac{dy}{dx} = \frac{1}{x}$

For $x < 0$,

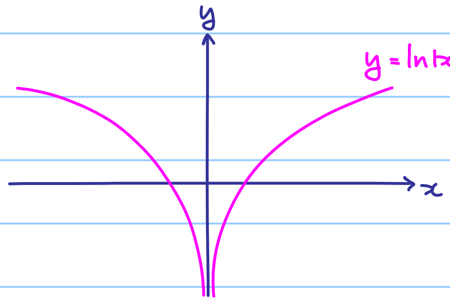
$$y = \ln(-x)$$

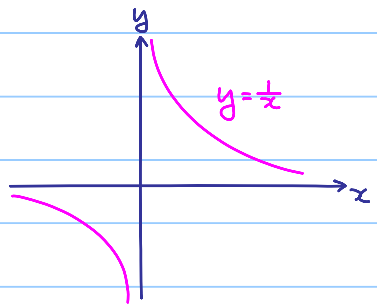
$$e^y = -x$$

$$e^y \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{e^y} = \frac{1}{x}$$

$\therefore \frac{d}{dx} \ln|x| = \frac{1}{x}$ for $x \neq 0$.





Note: It is why $\int \frac{1}{x} dx = \ln|x| + C$.

Example 5.8.5

Differentiation of Inverse Trigonometric Functions

Let $y = \sin^{-1} x$, $\sin^{-1} : [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, $\sin y = x$.

differentiate both sides with respect to x.

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$\sin y = x$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\cos y = \pm\sqrt{1-\sin^2 y}$

$= \sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

$\therefore \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

(rejected, $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$)

Let $y = \cos^{-1} x$, $\cos^{-1} : [-1,1] \longrightarrow [0,\pi]$. Then, $\cos y = x$.

differentiate both sides with respect to x.

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sin y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$\cos y = x$, $0 \le y \le \pi$

$\sin y = \pm\sqrt{1-\cos^2 y}$

$= \sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

$\therefore \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

(rejected, $0 \le y \le \pi \Rightarrow \sin y \ge 0$)

Exercise 5.8.1

Let $y = \tan^{-1} x$, $\tan^{-1} : \mathbb{R} \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.

Find $\frac{dy}{dx}$.   Ans: $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

Example 5.8.6

If $y = \sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}$, then find $\frac{dy}{dx}$.

Difficult to differentiate by using chain rule and quotient rule.

$$y^3 = \frac{(x-1)(x-2)^2}{x-4}$$

$$\ln y^3 = \ln \frac{(x-1)(x-2)^2}{x-4}$$

$$3 \ln y = \ln(x-1) + 2\ln(x-2) - \ln(x-4)$$

$$\frac{3}{y}\frac{dy}{dx} = \frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}$$

(Apply implicit differentiation)

$$\frac{dy}{dx} = \frac{y}{3}\left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}\right) = \frac{1}{3}\sqrt[3]{\frac{(x-1)(x-2)^2}{x-4}}\left(\frac{1}{x-1} + \frac{2}{x-2} - \frac{1}{x-4}\right)$$

Example 5.8.7

Let $y=\dfrac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$. Find $\dfrac{dy}{dx}$

$$y=\frac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$$

$$\ln y = 5x+\frac{1}{3}\ln(x^2+1)-4\ln(3x^2+1)$$

Ex: $\vdots$

Ans: $\dfrac{dy}{dx}=\left[5+\dfrac{2x}{3(x^2+1)}-\dfrac{24x}{3x^2+1}\right]\dfrac{e^{5x}\sqrt[3]{x^2+1}}{(3x^2+1)^4}$

Example 5.8.8

Let $y=x^x$, $x>0$. Find $\dfrac{dy}{dx}$.

Note: The power is NOT a constant, we cannot use the formular $\dfrac{d}{dx}x^n=nx^{n-1}$.

$$y=x^x$$

$$\ln y=\ln x^x=x\ln x$$

differentiate both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx}=\ln x+x\cdot\frac{1}{x}$$

$$=\ln x+1$$

$$\frac{dy}{dx}=(\ln x+1)y$$

$$=(\ln x+1)x^x$$

Example 5.8.9

Suppose $x^3+y^3-3xy=0$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

$$x^3+y^3-3xy=0$$

$$3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y-x^2}{y^2-x}$$

differentiate both sides with respect to x again.

$$\frac{d^2y}{dx^2}=\frac{\frac{dy}{dx}(y^2-x)-(y-x^2)(2y\frac{dy}{dx}-1)}{(y^2-x)^2}$$

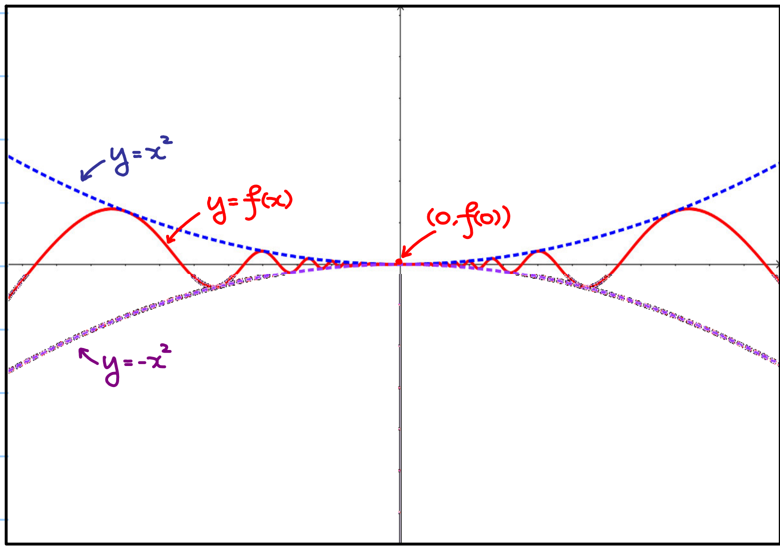
Sub. $\dfrac{dy}{dx}=\dfrac{y-x^2}{y^2-x}$ back to express $\dfrac{d^2y}{dx^2}$ in terms of x and y only, if you want.

Nightmare!

5.9 More on Differentiability

Example 5.9.1

Let $f(x)=\begin{cases} x^2\cos\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$



Does $f'(0)$ exist ?

$$\lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x^2\cos\frac{1}{\Delta x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\Delta x\cos\frac{1}{\Delta x}$$

$$=0$$

By sandwich theorem

If $x\neq 0$, $f'(x)=2x\cos\frac{1}{x}+x^2(-\sin\frac{1}{x})(-\frac{1}{x^2})$

$$=2x\cos\frac{1}{x}+\sin\frac{1}{x}$$

$\therefore$ f is a differentiable function, i.e. differentiable at every point.

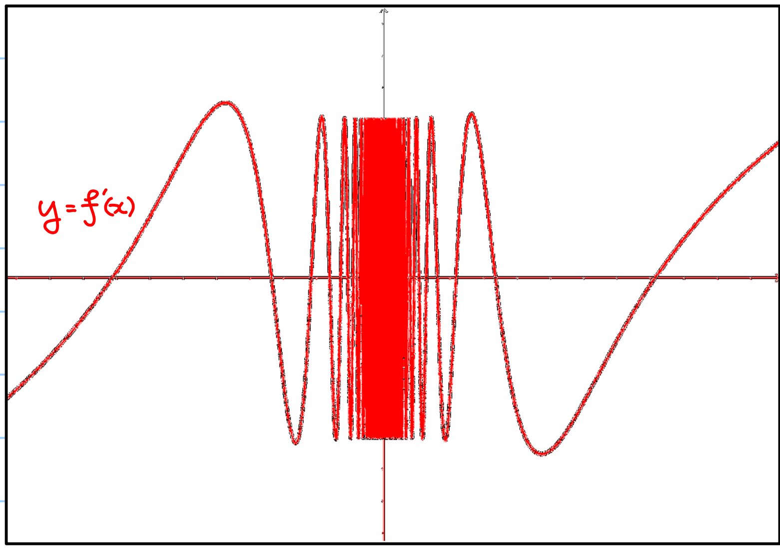
Note : It is wrong to say $f'(x)=2x\cos\frac{1}{x}+\sin\frac{1}{x}$, so $f'(0)$ does NOT exist .

Now, $f'(x)=\begin{cases} 2x\cos\frac{1}{x}+\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

Exercise :

Show $\lim_{x\to 0} f'(x)$ does NOT exist

($\Rightarrow$ $f'(x)$ is NOT continuous at $x=0$)



$\therefore$ f is differentiable ("good" in some sense)

but $f'(x)$ can be "bad".

Example 5.9.2

Let $f: \mathbb{R} \to \mathbb{R}$ be a non-constant function such that

(i) f is differentiable at some $x_0 \in \mathbb{R}$

(ii) $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Show that :

a) $f(x) \neq 0$ for all $x \in \mathbb{R}$ and $f(0) = 1$.

b) f is differentiable at every $x \in \mathbb{R}$ and $f'(x) = \frac{f'(x_0)}{f(x_0)} f(x)$.

a) If $f(a) = 0$ for some $a \in \mathbb{R}$

then for any $x \in \mathbb{R}$, we have

$$f(x) = f(x - a + a) = f(x-a)f(a) = 0$$

i.e. $f(x)$ is constant zero (Contradict to the assumption)

$\therefore f(x) \neq 0 \quad \forall x \in \mathbb{R}$.

Putting $x = y = 0$,

$$f(0+0) = f(0)f(0)$$

$$f(0) = [f(0)]^2$$

$$f(0) = 1 \quad \text{or} \quad 0 \text{ (rejected)}$$

b) f is differentiable at x_0

$$\Rightarrow f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0)f(\Delta x) - f(x_0)}{\Delta x}$$

$$\therefore \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = \frac{f'(x_0)}{f(x_0)} \qquad (\because f(x_0) \neq 0)$$

Now, $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{f(x)f(\Delta x) - f(x)}{\Delta x}$$

$$= f(x) \cdot \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$$

$$= \frac{f'(x_0)}{f(x_0)} f(x)$$

$\therefore f$ is differentiable at every $x \in \mathbb{R}$ and $f'(x) = \frac{f'(x_0)}{f(x_0)} f(x)$.

(In fact, $f(x) = e^{kx}$ for some non-zero constant k.)

Exercise 5.9.1

Let f be a differentiable function such that

$$f(x+y) = f(x) + f(y) + 3xy(x+y) \quad \forall\, x, y \in \mathbb{R}.$$

a) Show that $f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}$.

b) Hence, show that $f'(x) = f'(0) + 3x^2$.

(In fact, $f(x) = f'(0)x + x^3$.)